Skew plasticising component normal I distribution

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Abstract. This article has two goals. The first (main) goal is to introduce a new flexible distribution defined on an infinite domain $(-\infty, \infty)$. This distribution has been named the skew plasticising component normal distribution. The second (additional) goal is to present a chronological overview of distributions belonging to the large family of normal plasticising distributions. Some properties of the proposed distribution such as the PDF, CDF, quantiles, generator, moments, skewness, kurtosis and moments of order statistics are presented. The unknown parameters of the new distribution are estimated by means of the maximum likelihood method. The Shannon entropy, the Hessian Matrix and the Fisher Information Matrix are also presented. The study provides illustrative examples of the applicability and flexibility of the introduced distribution. The most important R codes are provided in Appendix 2.

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1. Introduction

The Gaussian distribution should be classified as a normal distribution (ND) due to the regularity and clarity of the roles of its parameters and its unique mathematical properties. However, it appears that its enormous popularity is disproportionate to its real applications. In many practical cases, empirical data exhibit skewness, heavy tails or multimodality that cannot be captured by the classical ND. The ND then needs to be plasticised.

As shown in numerous studies, various approaches have been developed to plasticise the ND, forming a broad family of normal plasticised distributions.

The relevant literature shows that there are various methods of plasticising the ND, forming a family of normal plasticising distributions.

The first group of normal plasticising distributions is a mixture of distributions, i.e. a mixture of a plasticising component and an ND. A mixture distribution, which is a combination of at least two distributions, can fit more characteristics than sample data might contain. Owing to this property, mixture distributions have been widely used in statistical sciences (Frühwirth-Schnatter, 2006;

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Martínez-Flórez et al., 2022). Behboodian (1970) presents a procedure for determining whether a mixture of two NDs (also called the compound normal (CN) distribution) is unimodal or not. Stephens (2000) studied what is called the 'label switching' problem, caused by the symmetry in the likelihood of the model parameters. A common response to this problem is to remove the symmetry by using artificial identifiability constraints. Lin et al. (2007) used a mixture of skew distribution models to fit multimodal data and datasets with bimodal features. Magnus and Magnus (2019) considered a subclass of the mixture models, namely normal latent factor mixture models. Popović et al. (2017) proposed the extended mixture ND, based on a linear mixture model, whose Probability Density Function (PDF) is symmetrical. Wang and Song (2017) developed a new equivalent linearisation method for nonlinear random vibration analysis. The method employs a Gaussian mixture distribution model to approximate the probabilistic distribution of a nonlinear system response. Sulewski (2022b) defined an ND with a plasticising component (NDPC).

The second group is a family of distributions with a plasticising formula located in the exponential function of the ND. This family includes e.g. the lognormal (Gaddum, 1945; Kapteyn, 1916), SL, SB, SU (Johnson, 1949), Birnbaum–Saunders (Athayde et al., 2012; Birnbaum & Saunders, 1969; Sulewski & Stoltmann, 2023), inverse Gaussian (Chhikara & Folks, 1977), sinh-normal (Rieck & Nedelman, 1991), DS normal (Sulewski, 2021), the Sulewski Plasticizing Component (Sulewski & Volodin, 2022), SC and SD (Sulewski, 2023) distributions.

The third group is a two-piece family of distributions. The PDFs of these distributions are in Table A1 (see Appendix 1). The family of distributions includes: the two-piece skew-normal (TPSN, Kim, 2005), generalised skew-normal (GSN1, Gómez et al., 2006), extended epsilon skew-normal (EESN, Salinas et al., 2007), epsilon skew normal (ESN, Mudholkar & Hutson, 2000), flexible epsilon-skew-normal (FESN, Arellano-Valle et al., 2010), skew-two-piece skew-normal (STPSN, Jamalizadeh & Arabpour, 2011), generalised two-piece skew-normal (GTPSN, Jamalizadeh & Arabpour, 2011), generalised skew-two-piece skew-normal (GSTPSN, Jamalizadeh & Arabpour, 2011), generalised two-piece skew-normal (GTPSN, Kumar & Anusree, 2013), two-piece power normal (TPPN, Sulewski, 2021) distributions.

The fourth group is a family of distributions with PDF $f(x;\boldsymbol{\theta})\phi(x)$, where the $f(x;\boldsymbol{\theta})$ is some function with parameter vector $\boldsymbol{\theta}$ and $\phi(x)$ is a PDF of $N(0,1)$. The PDFs of these distributions are presented in Table A1 in Appendix 1. The family of distributions includes the symmetric bimodal normal (BN, Arellano-Valle & Azzalini, 2008), alpha-skew-normal (ASN, Elal-Olivero, 2010), double normal (DN, Alavi, 2012), generalised alpha-skew-normal (GASN, Handam, 2012), Balakrishnan alpha-skew-normal (BASN, Hazarika et al., 2020), two-piece normal (TN, Salinas et al., 2023), alpha-beta skew-normal (ABSN, Shafiei et al., 2016), Balakrishnan alpha-beta-skew-normal (BABSN, Shah et al., 2021) and flexible alpha normal (FAN, Martínez-Flórez et al., 2022) distributions.

The fifth group is a family of distributions with PDF $f(x;\boldsymbol{\theta})exp(-|x|^{\gamma}/\gamma)$ $(\gamma > 0)$. The PDFs of these distributions are provided in Table A1 in Appendix 1. The family of distributions includes the generalised normal (GN, Kumar & Anusree, 2015), bimodal generalised normal (BGN, Mahmoudi et al., 2019) and alpha-skew generalised normal (ASGN, Mahmoudi et al., 2019) distributions.

The sixth group is a power normal family of distributions. The PDFs of these distributions are available in Table A1 in Appendix 1. The family of distributions includes the power normal (PN, Gupta & Gupta, 2008), generalised power-normal (GPN, Arnold et al., 2002), Durrans's power normal (Durrans, 1992) and power skew asymmetric normal (PSAN, Martínez-Flórez et al., 2014) distributions.

The seventh group is the Azzalini family of distributions. Azzalini (1985) added a skewness parameter to the Cumulative Distribution Function (CDF) of the ND and defined the skew-normal (SN) distribution with the following PDF:

$$f_{SN}(x;\lambda) = 2\phi(x)\Phi(\lambda\mathrm{x})\ (\lambda \in R), \tag{1}$$

where ϕ and Φ are the PDF and CDF of $N(0,1)$, respectively.

This distribution and its variations have been discussed by several authors including Azzalini (1985; 1986), Henze (1986), Azzalini & Dalla Valle (1996), Branco & Dey (2001), Loperfido (2001), Arnold et al. (2002) and Azzalini and Chiogna (2004). The PDFs of the Azzalini family of distributions are shown in Table A1 in Appendix 1. This family includes the skewed normal (SN1, Arnold et al., 2002), skew-curved normal (SCN, Arellano-Valle et al., 2004), skew-generalised normal (SGN, Arellano-Valle et al., 2004), flexible generalised skew-normal of order 3 (FGSN3, Ma & Genton, 2004), Balakrishnan skew–normal (BSN, Sharafi & Behboodian, 2008), generalised skew-normal (Gupta & Gupta, 2004), generalised skew-normal (GSN2, Jamalizadeh & Balakrishnan, 2008), two-parameter Balakrishnan skew-normal (TPBSN, Bahrami et al., 2009), generalised skew-normal (GSN, Jamalizadeh & Balakrishnan, 2008), skew bimodal normal (SBN) (Elal-Olivero et al., 2009), skew-flexible normal (SFN, Gómez et al., 2011), extended skew generalised normal I (ESGN1, Choudhury & Matin, 2011), extended skew generalised normal II (ESGN2, Choudhury & Matin, 2011), generalised mixture of standard normal and skew-normal (GMNSN, Kumar & Anusree, (2011), normal-skew-normal (NSN, Gómez et al., 2013), flexible skew-generalised normal (FSGN, Bahrami & Qasemi, 2015), flexible skew-curved normal (FSCN, Bahrami & Qasemi, 2015), extended skew generalised normal III (ESGN3, Kumar & Anusree, 2015), shape-skew-generalised normal (SSGN, Rasekhi et al., 2017), skew-bimodal normal-normal (SBNN, Alavi & Tarhani, 2017), extended skew-normal (ESN, Seijas-Macias et al., 2017), generalised alpha-beta skew-normal (GABSN, Shah et al., 2023) and flexible alpha-skew-normal (FASN, Das et al., 2023) distributions.

Despite this extensive literature, many existing plasticising models are either computationally demanding, lacking interpretability or they fail to simultaneously model skewness and bimodality in a parsimonious way. Motivated by these limitations, we introduce a new member of the normal plasticised distributions family, namely the skew plasticising component normal (SPCN1) distribution. This model provides a simple yet flexible way to generate a wide range of unimodal and bimodal shapes while preserving a clear probabilistic interpretation of its parameters.

The SPCN1 distribution extends the idea of a compound normal model by introducing a skew plasticising component that modifies both tails and the central concentration of the ND. The proposed formulation enables continuous control over skewness and kurtosis and allows the model to adapt to empirical data exhibiting asymmetric or bimodal behaviour. Furthermore, its analytical tractability makes it suitable for estimation via maximum likelihood and for use in simulation and goodness-of-fit studies.

This article has two goals. The first (main) goal is to introduce the SPCN1 distribution defined on an infinite domain $(-\infty, \infty)$. The second (additional) goal is to provide a chronological overview of distributions belonging to the large family of normal plasticising distributions.

This paper is organised as follows. Section 2 presents the properties of the SPCN1 distribution such as the PDF, CDF, quantiles, generator, moments, skewness, kurtosis and moments of order statistics. The Shannon entropy is presented in Section 3, while the Hessian Matrix and the Fisher Information Matrix are presented in Section 4. The maximum likelihood estimation is discussed in Section 5, while illustrative examples of the applicability and flexibility of the proposed distribution are presented in Section 6. The conclusions are presented in Section 7. The most important R codes are provided in Appendix 2. The PDFs of the large family of normal plasticising distributions are given in Table 1.

2. Properties of the proposed distribution

2.1. The probability density function

The PDF and CDF of the plasticising component (PC) are given (Sulewski, 2022b) by

$$f_{PC}(x;c) = \frac{c}{\sqrt{2\pi}}|x|^{c-1}exp\left[-\frac{1}{2}|x|^{2c}\right] = c|x|^{c-1}\phi(|x|^c), \tag{2}$$

$$F_{PC}(x;c) = \Phi[sgn(x)|x|^c], \tag{3}$$

where $c \geq 1$ is the shape parameter, and ϕ and Φ are the PDF and CDF of $N(0,1)$, respectively.

Definition 1. (the Azzalini transformation) The distribution of random variable X with the PDF given by

$$f(x;c,d) = 2f_{PC}(x;c)F_{PC}(xd;c) = 2c|x|^{c-1}\phi(|x|^c)\Phi[sgn(xd)|xd|^c] \tag{4}$$

or

$$f(x;c,d) = \frac{c|x|^{c-1}exp[-0.5|x|^{2c}]}{\sqrt{2\pi}}\left\{1 + erf\left[\frac{|xd|^c sgn(xd)}{\sqrt{2}}\right]\right\},$$

or

$$f(x;c,d) = \frac{c|x|^{c-1}exp[-0.5|x|^{2c}]}{\sqrt{2\pi}} erfc\left[\frac{-|xd|^c sgn(xd)}{\sqrt{2}}\right],$$

or

$$f(x;c,d) = \frac{c}{\sqrt{2\pi}}\begin{cases} x^{c-1}exp[-0.5x^{2c}]erfc\left[\frac{-(xd)^c}{\sqrt{2}}\right], x \geq 0 \\ (-x)^{c-1}exp[-0.5(-x)^{2c}]erfc\left[\frac{(-xd)^c}{\sqrt{2}}\right], x < 0 \end{cases}$$

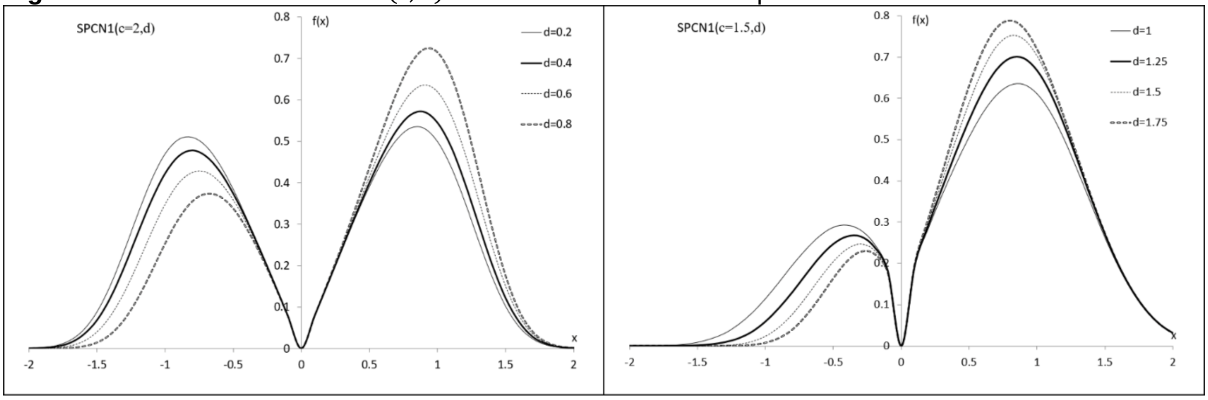
is called the skew plasticising component normal I (SPCN1) distribution, where $c \geq 1$ is the shape parameter, $d \geq 0$ is the skewness parameter, $erf(.)$ is the error function and $erfc(.)$ is the complementary error function. For $c = 1, d = 0$, we obtain the $N(0,1)$ and for $d = 0$, we obtain the PC (2). The symbol ‘I’ denotes the authors' first proposal for the skew plasticising component normal (SPCN) distribution. The R codes of the dSPCN1 function are presented in Appendix 2.

Figure 1 shows the PDF of the $\mathrm{SPCN1(c,d)}$ for some values of the parameters. If $c > 1$, the PDF has two modes of various heights.

The $SP\mathrm{CN1(c,d)}$ can be used to deviate from the $N(0,1)$. The similarity measure between our proposal and the $N(0,1)$ was provided by Sulewski (2022a):

$$M(c,d) = \int_{-\infty}^{\infty} min[f(x;c,d), \phi(x)]dx. \quad (5)$$

Figure 1. The PDF of the $\mathrm{SPCN1}(c,d)$ for selected values of the parameters



Source: authors' work.

As mentioned before, the $SP\mathrm{CN}1(1{,}0)$ is the $N(0{,}1)$, so $M_{max}(1{,}0) = 1$.

In addition to similarity measure M, the difference between the distributions can also be quantified using the Kullback–Leibler (KL) divergence. For two PDFs, p and q, the KL divergence is defined as (Kullback & Leibler, 1951)

$$KL(p,q) = \int_{-\infty}^{\infty} p(x;\boldsymbol{\theta_p}) log_2 \frac{p(x;\boldsymbol{\theta_p})}{q(x;\boldsymbol{\theta_q})} dx, \quad (6)$$

where $\boldsymbol{\theta_p}$ is the parameter vector of function $p(x)$, $\boldsymbol{\theta_q}$ is the parameter vector of function $q(x)$. KL takes the values of $(0, \infty)$.

In our context, the KL is given by

$$KL(f,\phi) = \int_{-\infty}^{\infty} f(x;c,d) log_2 \frac{f(x;c,d)}{\phi(x)} dx.$$

Tables 1 and 2 summarise similarity measure M and the complement of the Kullback–Leibler divergence $1 - KL$ between the $SPCN1$ distribution and standard ND $N(0{,}1)$. The KL measure has been written as $1 - KL$ to make it easier to compare with the M similarity measure.

In both cases, as parameters c and d depart from their reference values ($c = 1$ and $d = 0$), similarity measure M decreases, indicating a gradual divergence from the ND.

The values of $1 - KL$ show a consistent trend with M: as d or c increases, divergence KL between $SPCN1$ and $N(0{,}1)$ grows. For small deviations of c and d, both measures suggest a strong resemblance. For larger parameter values, the $SPCN1$ distribution becomes increasingly non-normal, as evidenced by the steep decline of both measures.

In particular, Table 1 illustrates that skewness parameter d has a strong influence on similarity: even moderate departures from $d = 0$ lead to a noticeable drop in both M and $1 - KL$. Table 2 shows a similar, but slightly smoother effect for shape parameter c.

Overall, both measures (M and KL) provide consistent quantitative evidence that $SPCN1(c, d)$ continuously and controllably deviates from $N(0,1)$, confirming its flexibility and interpretability as a 'plasticized' version of the ND.

Table 1. Similarity measure $M(1, d)$ and $KL(1, d)$ between the SPCN1$(1, d)$ and $N(0,1)$

d	0	0.158	0.325	0.51	0.727	1	1.376	1.963	3.078	6.314	29.82
$M(1, d)$	1	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5
$1 - KL(1, d)$	1	0.989	0.955	0.898	0.82	0.721	0.603	0.468	0.32	0.162	0.035

Source: authors' work.

Table 2. Similarity measure $M(c, 0)$ and $KL(c, 0)$ between the SPCN1(c, 0) and $N(0,1)$

c	1	1.118	1.253	1.404	1.576	1.775	2.005	2.278	2.602	2.991	3.469
$M(c, 0)$	1	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5
$1 - KL(c, 0)$	1	0.985	0.945	0.885	0.807	0.712	0.602	0.474	0.331	0.172	−0.006

Source: authors' work.

2.2. Cumulative density function

Let $X \sim \text{SPCN1}(c, d)$. The CDF of the SPCN1 distribution, based on definition 1, is given by the following formula:

$$F(x; c, d) = 2c \int_{-\infty}^{x} |t|^{c-1} \phi(|t|^c) \Phi[sgn(td)|td|^c] dt. \tag{7}$$

For $x < 0$, formula (7) can be written as

$$F(x; c, d) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{x} \frac{(-t)^{c-1}}{e^{0.5(-t)^{2c}}} erfc\left[\frac{(-td)^c}{\sqrt{2}}\right] dt$$

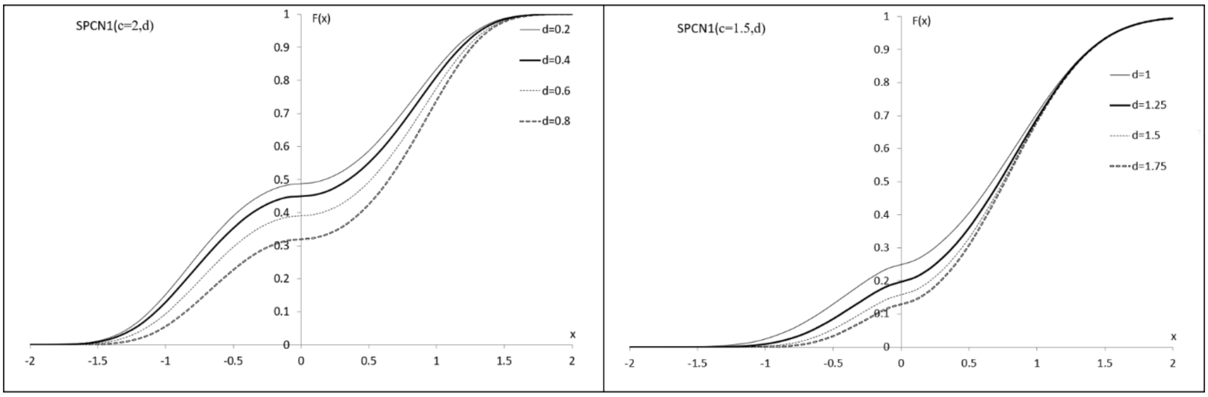
and for $x \geq 0$, we have

$$F(x; c, d) = \frac{c}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} \frac{(-t)^{c-1}}{e^{0.5(-t)^{2c}}} erfc\left[\frac{(-td)^c}{\sqrt{2}}\right] dt + \int_{0}^{x} \frac{t^{c-1}}{e^{0.5t^{2c}}} erfc\left[\frac{-(td)^c}{\sqrt{2}}\right] dt \right\}.$$

The R codes of the pSPCN1 function are presented in Appendix 2.

Figure 2 shows the CDF of the SPCN1(c, d) for some parameter values. For $c > 1$, we obtain two sub-CDFs placed at certain levels, which means the distribution is bimodal.

Figure 2. The CDF of the SPCN1(c, d) for some values of the parameters



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It is quite understandable that the CDF does not have a closed form, since the distribution in question has its origin in the Gaussian distribution. A similar situation, as can be seen below, concerns quantiles, the pseudo-random number generator, non-central moments, moments of order statistics, and the Shannon entropy. However, this is not a problem from the perspective of practical applications, because thanks to numerical methods, we obtain user functions written, for example in the R environment (see Appendix 2).

2.3. Quantile and pseudo-random number generator

Let $X \sim \text{SPCN1}(c,d)$. The p-th $(0 < p < 1)$ quantile x_p is a solution to equation

$$\frac{c}{\sqrt{2\pi}} \int_{-\infty}^{x_p} \frac{|x|^{c-1}}{e^{0.5|x|^{2c}}} erfc\left[\frac{-|xd|^c sgn(xd)}{\sqrt{2}}\right] dx - p = 0. \tag{8}$$

The R codes of the qSPCN1 function are presented in Appendix 2.
Let $X \sim \text{SPCN1}(c,d)$ and $R \sim Unif(0,1)$. The pseudo-random number generator of X is a solution to equation

$$\frac{c}{\sqrt{2\pi}} \int_{-\infty}^{X} \frac{|x|^{c-1}}{e^{0.5|x|^{2c}}} erfc\left[\frac{-|xd|^c sgn(xd)}{\sqrt{2}}\right] - R = 0. \tag{9}$$

The R codes of the rSPCN1 function are presented in Appendix 2.

2.4. Moments

Let $X \sim \text{SPCN1}(c,d)$. Non-central moments of X are given by

$$\alpha_k(c,d) = E(X^k) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{|x|^{c-1}}{e^{0.5|x|^{2c}}} erfc\left[\frac{-|xd|^c sgn(xd)}{\sqrt{2}}\right] dx \;\; (k = 1,2,\dots). \tag{10}$$

The R codes of the mSPCN1 function are presented in Appendix 2.

2.5. Skewness and kurtosis

Based on the order (non-central) moments and using their relationships with central moments $\mu_k = \sum_{i=0}^{k}(-1)^i \binom{k}{i} \alpha_{k-i}\alpha_1^i$, we can easily calculate skewness γ_1 and kurtosis γ_2 of the $SPCN1(c,d)$.

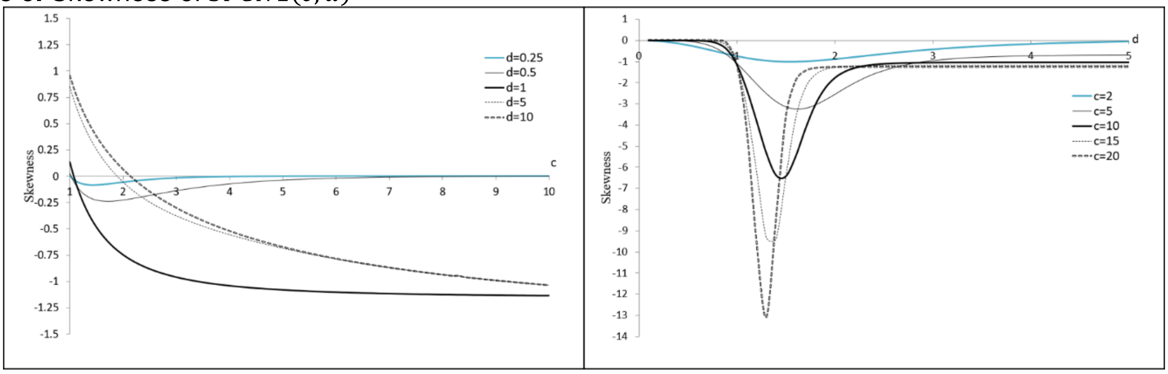
The skewness of $\mathrm{SPCN1}(c,d)$ is defined as

$$\gamma_1(c,d) = \frac{\mu_3}{\mu_2^{1.5}} = \frac{\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3}{(\alpha_2 - \alpha_1^2)^{1.5}},$$

where α_i $(i = 1,2,3)$ are given by (10). The R codes of the g1SPCN1 function are presented in Appendix 2.

Figure 3 shows γ_1 as a function of c for selected d values (left) and γ_1 as a function of d for selected c values (right). $\gamma_1(c)$ is a decreasing function for $d \geq 1$, especially for the initial values of the arguments and inversely unimodal for $0 < d < 1$, e.g. $\gamma_1^{min}(1.713{,}0.5) = -0.239$. As d increases, $\gamma_1(c)$ decreases. $\gamma_1(d)$ is inversely unimodal for $c \geq 1$ e.g. $\gamma_1^{min}(10{,}1.455) = -6.54$. The $\gamma_1(d)$ function is strictly monotonical for the initial values of the arguments.

Figure 3. Skewness of $SPCN1(c,d)$



Source: authors' work.

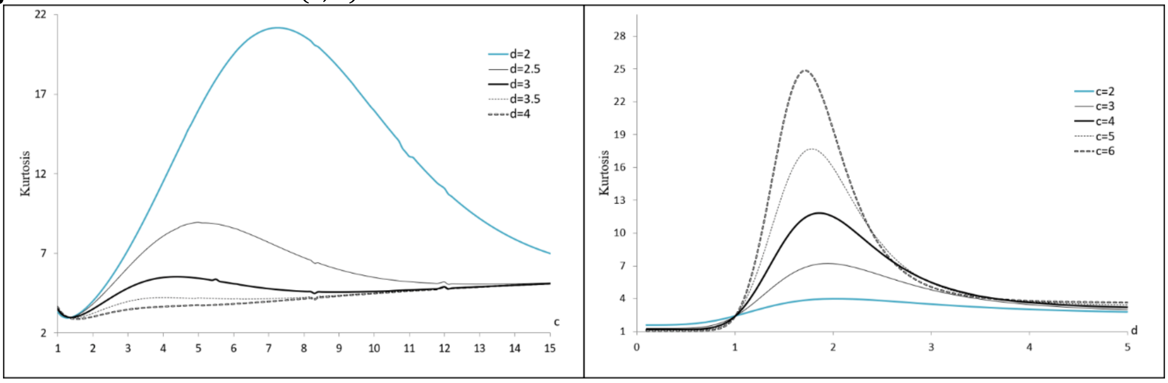
The kurtosis of $\mathrm{SPCN1}(c,d)$ is given by

$$\gamma_2(c,d) = \frac{\mu_4}{\mu_2^2} = \frac{\alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4}{(\alpha_2 - \alpha_1^2)^2},$$

where α_i $(i = 1,2,\dots,4)$ are given by (10). The R codes of the g2SPCN1 function are presented in Appendix 2.

Figure 4 shows γ_2 as a function of c for selected d values (left) and γ_2 as a function of d for selected c values (right). $\gamma_2(c)$ is the unimodal function, e.g. $\gamma_2^{max}(7.253,2) = 21.162$. As d increases, $\gamma_2(c)$ decreases. $\gamma_2(d)$ is the unimodal function, e.g. $\gamma_2^{max}(6,1.716) = 24.87$. As c increases, $\gamma_2(c)$ also increases. As Malakhov's inequality $\gamma_2 \geq \gamma_1^2 + 1$ (Malakhov, 1978) indicates, we obtain γ_2 equal to no less than 1.

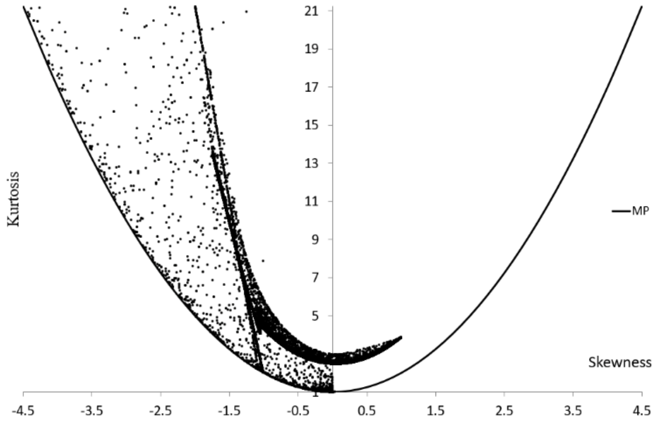
Figure 4. Kurtosis of $SPCN1(c,d)$



Source: authors' work.

We calculate γ_1 and γ_2 for 10^5 random values of $c = Unif(1,100)$ and $d = Unif(0,100)$. Figure 5 presents a set of points (γ_1, γ_2) located in a rectangle $(-4.5,4.5) \times (1,21.25)$. The symbol MP denotes the $\gamma_2 = \gamma_1^2 + 1$ Malakhov parabola. We obtain $\gamma_1 \in (-4.824,0.994)$, $\gamma_2 \in (1,21.244)$ and a very interesting shape.

Figure 5. Variability range of γ_1 and γ_2 of $SPCN1(c,d)$



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2.6. Moments of order statistics

Let $X_{i,n}$ be the i-th order statistic $(X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n})$ in a sample of size n from the SPCN1(c,d). The k-th moment of the i-th order statistic, $X_{i,n}$ is defined as

$$\alpha_{k,i,n} = E\left(\mathrm{X}_{i,n}^k\right) = \int_{-\infty}^{\infty} x^k f_{i,n}(x;a,b)dx, \tag{11}$$

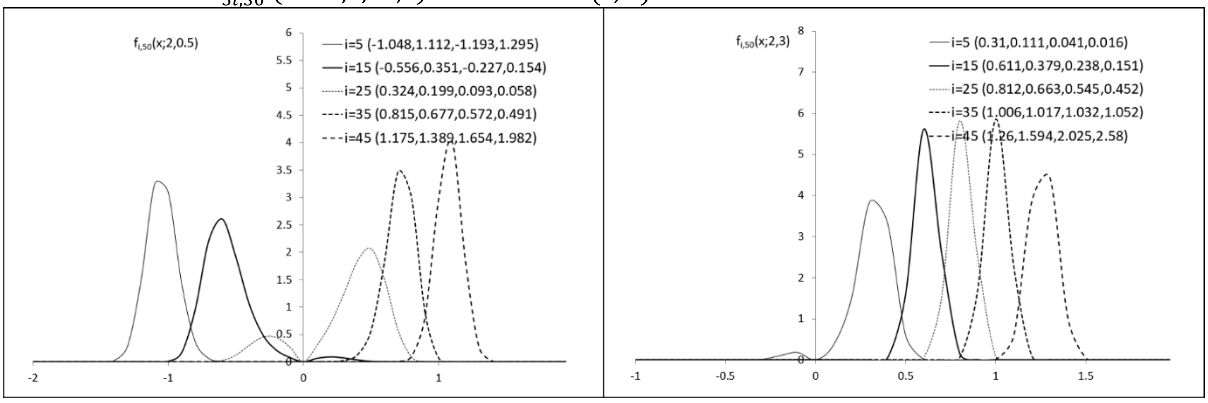
where

$$f_{i,n}(x; c, d) = i! \binom{n}{i} \frac{f(x;c,d)}{F(x;a,b)^{1-i}} [1 - F(x; a, b)]^{n-i} \tag{12}$$

and $f(x; a, b), F(x; a, b)$ are given by (4) and (7). The R codes of the mOSSPCN1 function are presented in Appendix 2. Note that from (12), we have $f_{2,2}(x; c, d) = 2f(x; c, d)F(x; a, b)$, so we obtain the Azzalini transformation without the skewness parameter.

Figure 6 shows the PDF of the $X_{5i,30}$ $(i = 1, 2, 3, 4, 5)$ of the $SPCN1(2, 0.5)$ (left) and $SPCN1(2, 3)$ (right), as well as $\alpha_{k,i,n}(k = 1, 2, 3, 4)$ in brackets, respectively. The $f_{i,50}(x_m; a, b)$ value is the highest for $i = 45$ (Figure 6, left) and for $i = 35$ (Figure 6, right). The values of $\alpha_{k=1,i,n}$ and $\alpha_{k=3,i,n}$ increase along with the i value.

Figure 6. PDF of the $X_{5i,30}$ $(i = 1,2, \dots ,5)$ of the $SPCN1(c, d)$ distribution



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3. Shannon entropy

Let $f(x; c, d)$ be the PDF (4). Shannon entropy S is given by (Shannon, 1948)

$$S(c, d) = -\int_{-\infty}^{\infty} f(x; c, d) ln[f(x; c, d)]dx, \tag{13}$$

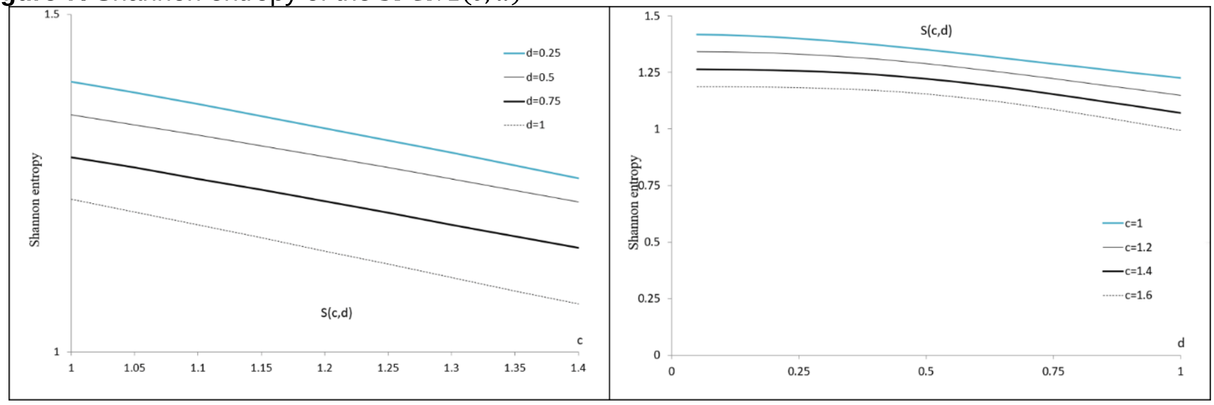
where

$$ln[f(x; c, d)] = nln \frac{c}{\sqrt{2\pi}} + (c - 1)ln|x| - 0.5|x|^{2c} + ln \left\{erfc \left[\frac{-|xd|^c sgn(xd)}{\sqrt{2}}\right]\right\}. \tag{14}$$

The R codes of the sSPCN1 function are presented in Appendix 2.

Figure 7 shows the Shannon entropy as a function of c for selected d values (left) and as a function of d for selected c values (right). We obtain decreasing functions with a very small numerical range of variability.

Figure 7. Shannon entropy of the $SPCN1(c,d)$



Source: authors' work.

4. Hessian Matrix and Fisher Information Matrix

The Hessian Matrix (HM) and the Fisher Information Matrix (FIM) are critical for optimisation and statistical inference. Since the HM is related to the FIM, its singularity (non-invertibility) indicates that the log-likelihood function may have flat regions or an insufficient curvature at the given parameter values. This lack of curvature can lead to the FIM, which quantifies the precision of parameter estimates, being singular or near-singular.

The FIM quantifies the amount of information that a random variable stores about an unknown parameter. Let $f(x;c,d)$ and $ln[f(x;c,d)]$ be given by (4) and (14), respectively. If there are suitable partial derivatives of $f(x;c,d)$, then the FIM $I_{i,j}^{c,d}(i,j=1,2)$ is a square 2×2 matrix defined as

$$I_{i,j}^{c,d} = -\begin{bmatrix} E\left\{\frac{\partial^2 ln[f(x;c,d)]}{dc^2}\right\} & E\left\{\frac{\partial^2 ln[f(x;c,d)]}{\partial c \partial d}\right\} \\ E\left\{\frac{\partial^2 ln[f(x;c,d)]}{\partial d \partial c}\right\} & E\left\{\frac{\partial^2 ln[f(x;c,d)]}{\partial d^2}\right\} \end{bmatrix}, \quad (15)$$

where $I_{1,2}^{c,d} = I_{2,1}^{cd}$, obviously. The R codes of the fimSPCN1 function are presented in Appendix 2.

A non-invertible (singular) HM can lead to the information matrix becoming singular, impacting the optimisation process and parameter estimation. If all second-order partial derivatives of $f(x;c,d)$ exist, then HM $H_{i,j}^{c,d}(i,j=1,2)$ is a square 2×2 matrix arranged as

$$H_{i,j}^{c,d} = \begin{bmatrix} \frac{\partial^2 f(x;c,d)}{\partial c^2} & \frac{\partial^2 f(x;c,d)}{\partial c \partial d} \\ \frac{\partial^2 f(x;c,d)}{\partial d \partial c} & \frac{\partial^2 f(x;c,d)}{\partial d^2} \end{bmatrix}, \tag{16}$$

where $H_{1,2}^{c,d} = H_{2,1}^{cd}$, obviously. The R codes of the hmSPCN1 function are presented in Appendix 2.

In distribution theory, there are papers with more or less complicated FIM and HM formulas, but it is difficult to find a numerical analysis.

The values of $I_{i,j}^{c,d}$ $(i,j = 1,2)$ for certain parameter values, including those from Figure 1, are:

$$I_{i,j}^{1,0.2} = \begin{bmatrix} 1.78 & -0.15 \\ -0.15 & 0.61 \end{bmatrix}, I_{i,j}^{1,0.4} = \begin{bmatrix} 1.79 & -0.13 \\ -0.13 & 0.54 \end{bmatrix}, I_{i,j}^{1,0.6} = \begin{bmatrix} 1.78 & -0.07 \\ -0.07 & 0.44 \end{bmatrix}, I_{i,j}^{1,0.8} = \begin{bmatrix} 1.79 & -0.02 \\ -0.02 & 0.35 \end{bmatrix},$$

$$I_{i,j}^{1.5,0.2} = \begin{bmatrix} 0.78 & -0.05 \\ -0.05 & 0.15 \end{bmatrix}, I_{i,j}^{1.5,0.4} = \begin{bmatrix} 0.79 & -0.10 \\ -0.10 & 0.53 \end{bmatrix}, I_{i,j}^{1.5,0.6} = \begin{bmatrix} 0.79 & -0.09 \\ -0.09 & 0.69 \end{bmatrix}, I_{i,j}^{1.5,0.8} = \begin{bmatrix} 0.79 & -0.03 \\ -0.03 & 0.70 \end{bmatrix},$$

$$I_{i,j}^{2,0.2} = \begin{bmatrix} 0.43 & -0.01 \\ -0.01 & 0.10 \end{bmatrix}, I_{i,j}^{2,0.4} = \begin{bmatrix} 0.44 & -0.06 \\ -0.06 & 0.40 \end{bmatrix}, I_{i,j}^{2,0.6} = \begin{bmatrix} 0.45 & -0.08 \\ -0.08 & 0.8 \end{bmatrix}, I_{i,j}^{2,0.8} = \begin{bmatrix} 0.45 & -0.05 \\ -0.05 & 1.09 \end{bmatrix},$$

$$I_{i,j}^{1,1} = \begin{bmatrix} 1.80 & 0.02 \\ 0.02 & 0.27 \end{bmatrix}, I_{i,j}^{1,1.25} = \begin{bmatrix} 1.81 & 0.04 \\ 0.04 & 0.19 \end{bmatrix}, I_{i,j}^{1,1.5} = \begin{bmatrix} 1.82 & 0.05 \\ 0.05 & 0.14 \end{bmatrix}, I_{i,j}^{1,1.75} = \begin{bmatrix} 1.83 & 0.05 \\ 0.05 & 0.10 \end{bmatrix},$$

$$I_{i,j}^{1.25,1} = \begin{bmatrix} 1.15 & 0.02 \\ 0.02 & 0.42 \end{bmatrix}, I_{i,j}^{1.25,1.25} = \begin{bmatrix} 1.16 & 0.05 \\ 0.05 & 0.31 \end{bmatrix}, I_{i,j}^{1.25,1.5} = \begin{bmatrix} 1.17 & 0.06 \\ 0.06 & 0.21 \end{bmatrix}, I_{i,j}^{1.25,1.75} = \begin{bmatrix} 1.17 & 0.06 \\ 0.06 & 0.15 \end{bmatrix},$$

$$I_{i,j}^{1.5,1} = \begin{bmatrix} 0.80 & 0.02 \\ 0.02 & 0.61 \end{bmatrix}, I_{i,j}^{1.5,1.25} = \begin{bmatrix} 0.81 & 0.06 \\ 0.06 & 0.44 \end{bmatrix}, I_{i,j}^{1.5,1.5} = \begin{bmatrix} 0.81 & 0.06 \\ 0.06 & 0.31 \end{bmatrix}, I_{i,j}^{1.5,1.75} = \begin{bmatrix} 0.81 & 0.06 \\ 0.06 & 0.21 \end{bmatrix}.$$

We obtain positive values of $I_{i,j}^{c,d}$ $(i,j = 1,2)$ except for values $I_{1,2}^{c,d} = I_{2,1}^{c,d}(d < 1)$. If $c = const$ and values of d increase, then values of $I_{2,2}^{c,d}(d \geq 1)$ decrease. If $d = const$ and values of c increase, then values of $I_{1,1}^{c,d}$ decrease, values of $I_{1,2}^{c,d} = I_{2,1}^{c,d}$ are similar and values of $I_{2,2}^{c,d}(d \geq 1)$ increase.

The values of $H_{i,j}^{c,d}$ $(i,j = 1,2)$ for certain parameter values, including those from Figure 1, are:

$$H_{i,j}^{1,0.2} = \begin{bmatrix} -0.19 & -0.17 \\ -0.17 & -0 \end{bmatrix}, H_{i,j}^{1,0.4} = \begin{bmatrix} -0.15 & -0.13 \\ -0.13 & -0 \end{bmatrix}, H_{i,j}^{1,0.6} = \begin{bmatrix} -0.14 & -0.10 \\ -0.10 & -0 \end{bmatrix}, H_{i,j}^{1,0.8} = \begin{bmatrix} -0.14 & -0.08 \\ -0.08 & -0 \end{bmatrix},$$

$$H_{i,j}^{1.5,0.2} = \begin{bmatrix} 0.13 & -0.04 \\ -0.04 & 0.03 \end{bmatrix}, H_{i,j}^{1.5,0.4} = \begin{bmatrix} 0.16 & -0.05 \\ -0.05 & 0.02 \end{bmatrix}, H_{i,j}^{1.5,0.6} = \begin{bmatrix} 0.18 & -0.05 \\ -0.05 & 0.02 \end{bmatrix}, H_{i,j}^{1.5,0.8} = \begin{bmatrix} 0.20 & -0.05 \\ -0.05 & 0.02 \end{bmatrix},$$

$$H_{i,j}^{2,0.2} = \begin{bmatrix} 0.16 & -0.01 \\ -0.01 & 0.01 \end{bmatrix}, H_{i,j}^{2,0.4} = \begin{bmatrix} 0.16 & -0.01 \\ -0.01 & 0.01 \end{bmatrix}, H_{i,j}^{2,0.6} = \begin{bmatrix} 0.17 & -0.02 \\ -0.02 & 0.01 \end{bmatrix}, H_{i,j}^{2,0.8} = \begin{bmatrix} 0.18 & -0.02 \\ -0.02 & 0.01 \end{bmatrix},$$

$$H_{i,j}^{1,1} = \begin{bmatrix} -0.15 & -0.07 \\ -0.07 & -0 \end{bmatrix}, H_{i,j}^{1,1.25} = \begin{bmatrix} -0.18 & -0.05 \\ -0.05 & -0 \end{bmatrix}, H_{i,j}^{1,1.5} = \begin{bmatrix} -0.2 & -0.04 \\ -0.04 & -0 \end{bmatrix}, H_{i,j}^{1,1.75} = \begin{bmatrix} -0.24 & -0.03 \\ -0.03 & -0 \end{bmatrix},$$

$$H_{i,j}^{1.25,1} = \begin{bmatrix} 0.13 & -0.07 \\ -0.07 & 0 \end{bmatrix}, H_{i,j}^{1.25,1.25} = \begin{bmatrix} 0.14 & -0.06 \\ -0.06 & 0 \end{bmatrix}, H_{i,j}^{1.25,1.5} = \begin{bmatrix} 0.14 & -0.05 \\ -0.05 & 0 \end{bmatrix}, H_{i,j}^{1.25,1.75} = \begin{bmatrix} 0.13 & -0.05 \\ -0.05 & 0 \end{bmatrix},$$

$$H_{i,j}^{1.5,1} = \begin{bmatrix} 0.21 & -0.05 \\ -0.05 & 0.01 \end{bmatrix}, H_{i,j}^{1.5,1.25} = \begin{bmatrix} 0.23 & -0.05 \\ -0.05 & 0.01 \end{bmatrix}, H_{i,j}^{1.5,1.5} = \begin{bmatrix} 0.24 & -0.05 \\ -0.05 & 0.01 \end{bmatrix}, H_{i,j}^{1.5,1.75} = \begin{bmatrix} 0.25 & -0.05 \\ -0.05 & 0.01 \end{bmatrix},$$

We get $H_{1,2}^{c,d} = H_{2,1}^{c,d} < 0$. Values of $H_{2,2}^{c,d}$ are very close to zero, e.g. $H_{2,2}^{1,0.2} = -0.0004988119$, $H_{2,2}^{1,0.4} = -0.0009952323$, $H_{2,2}^{1,0.6} = -0.001486889$. If $d = const$ and values of c increase then values of $H_{1,2}^{c,d}(d < 1)$ increase, values of $H_{1,2}^{c,d} = H_{2,1}^{c,d}$ are similar and values of $H_{1,2}^{c,d} = H_{2,1}^{c,d}(d < 1)$ and $H_{1,1}^{c,d}(d \geq 1)$ increase.

5. Maximum likelihood estimation

In this section, we present a location-scale SPCN1 distribution characterised by location parameter $\mu \in R$ and scale parameter $\sigma > 0$. This distribution is formulated through the $Y = \mu + \sigma X$ transformation:

$$f(y; \mu, \sigma, c, d) = \frac{2c}{\sigma} \left|\frac{y-\mu}{\sigma}\right|^{c-1} \phi\left(\left|\frac{y-\mu}{\sigma}\right|^{c}\right) \Phi\left[sgn\left(d\frac{y-\mu}{\sigma}\right)\left|d\frac{y-\mu}{\sigma}\right|^{c}\right]. \tag{17}$$

Let $y_1^*, y_2^*, \ldots, y_n^*$ be a random sample of size n from the $SPCN1(\mu, \sigma, c, d)$. Our target is to estimate the unknown μ, σ, c, d parameters. The likelihood function based on (2) is given by

$$L = \frac{2c}{\sigma} \prod_{i=1}^{n} \left|\frac{y_i^*-\mu}{\sigma}\right|^{c-1} \phi\left(\left|\frac{y_i^*-\mu}{\sigma}\right|^{c}\right) \Phi\left[sgn\left(d\frac{y_i^*-\mu}{\sigma}\right)\left|d\frac{y_i^*-\mu}{\sigma}\right|^{c}\right], \tag{18}$$

then the log-likelihood function $l = lnL$ is defined as:

$$l = nln\frac{2c}{\sigma} + (c-1)\sum_{i=1}^{n} ln\left|\frac{y_i^*-\mu}{\sigma}\right| + \sum_{i=1}^{n} ln\left[\phi\left(\left|\frac{y_i^*-\mu}{\sigma}\right|^{c}\right)\right] + \sum_{i=1}^{n} ln\left\{\Phi\left[sgn\left(d\frac{y_i^*-\mu}{\sigma}\right)\left|d\frac{y_i^*-\mu}{\sigma}\right|^{c}\right]\right\}. \tag{19}$$

There is no need to present formulas $\frac{dl}{d\mu}, \frac{dl}{d\sigma}, \frac{dl}{dc}, \frac{dl}{dd}$, because they have a complicated form. To simplify the process, we can use one of the advanced computational environments with embedded optimisation procedures. These include Mathcad, Mathematica, Excel or R. For the purpose of this paper, the maximum likelihood estimates (MLEs) of the μ, σ, c, d parameters were calculated in the R environment.

For estimated parameter Θ, the bias (BIAS) of estimator $\hat{\Theta}$ is defined as

$$BIAS(\hat{\Theta}) = \frac{1}{n}\sum_{i=1}^{n} \hat{\Theta}(y_1^*, y_2^*, \ldots, y_n^*) - \Theta$$

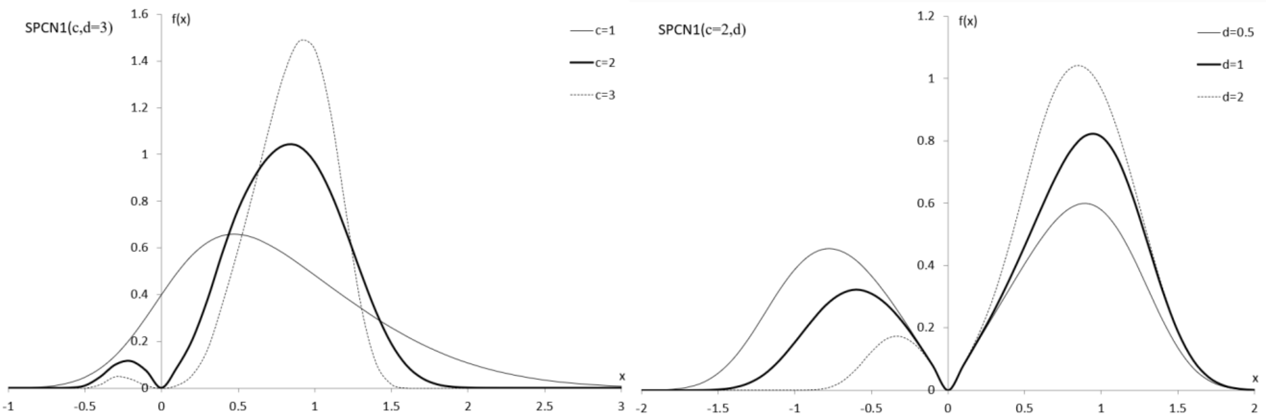
and the root mean squared error (RMSE) is given by

$$RMSE(\hat{\Theta}) = \sqrt{E\left[\left(\hat{\Theta} - \Theta\right)^2\right]}.$$

These characteristics of the MLEs are shown in Tables 3 and 4. The simulation study was performed with 10^3 samples using sample sizes of $25, 50, 100, 200$. The samples, as shown in Figure 8, were drawn from the $SPCN1(c, 3)$, $c = (1, 2, 3)$ and $SPCN1(2, d)$, $d = (0.5, 1, 2)$. Our

MLE analysis is for a unimodal and slightly bimodal distribution (left) as well as a clearly bimodal distribution (right).

Figure 8. PDF curves of the SPCN1 distribution for parameter values used in the MLE



Source: authors' work.

Table 3. Biases and RMSEs of the MLEs from $SPCN1(0,1,c,3)$

		$\hat{\mu}$		$\hat{\sigma}$		$\hat{c}$		$\hat{d}$	
c	n	BIAS	RMSE	BIAS	RMSE	BIAS	RMSE	BIAS	RMSE
1	25	-0.087	0.028	0.117	0.028	0.202	0.069	0.327	0.125
	50	-0.076	0.019	0.104	0.021	0.154	0.043	0.316	0.119
	100	-0.062	0.011	0.086	0.014	0.105	0.022	0.313	0.116
	200	-0.053	0.007	0.073	0.009	0.079	0.012	0.298	0.108
2	25	-0.065	0.019	0.087	0.018	0.265	0.099	0.279	0.100
	50	-0.052	0.011	0.071	0.011	0.218	0.075	0.270	0.096
	100	-0.038	0.006	0.058	0.007	0.173	0.049	0.266	0.092
	200	-0.024	0.002	0.043	0.003	0.124	0.025	0.238	0.078
3	25	-0.055	0.011	0.066	0.011	0.280	0.105	0.275	0.097
	50	-0.045	0.007	0.058	0.007	0.262	0.095	0.272	0.095
	100	-0.036	0.004	0.048	0.005	0.222	0.071	0.262	0.090
	200	-0.030	0.002	0.041	0.003	0.194	0.055	0.243	0.080

Source: authors' work.

Table 4. Biases and RMSEs of the MLEs from $SPCN1(0,1,2,d)$

		$\hat{\mu}$		$\hat{\sigma}$		$\hat{c}$		$\hat{d}$	
d	n	BIAS	RMSE	BIAS	RMSE	BIAS	RMSE	BIAS	RMSE
0.5	25	-0.023	0.008	0.048	0.005	0.226	0.076	0.172	0.051
	50	-0.017	0.005	0.043	0.003	0.188	0.056	0.143	0.034
	100	-0.013	0.002	0.035	0.002	0.158	0.039	0.121	0.023
	200	-0.007	0.001	0.033	0.002	0.139	0.030	0.101	0.016
1	25	-0.043	0.011	0.068	0.010	0.211	0.070	0.182	0.055
	50	-0.032	0.006	0.058	0.006	0.182	0.053	0.150	0.039
	100	-0.020	0.003	0.047	0.004	0.140	0.033	0.115	0.023
	200	-0.016	0.002	0.041	0.002	0.121	0.024	0.098	0.015
2	25	-0.050	0.012	0.076	0.013	0.242	0.086	0.263	0.096
	50	-0.040	0.007	0.064	0.008	0.206	0.066	0.243	0.084
	100	-0.031	0.004	0.053	0.005	0.162	0.044	0.212	0.068
	200	-0.023	0.002	0.045	0.003	0.127	0.027	0.187	0.052

Source: authors' work.

Tables 3 and 4 summarise the simulation results for the bias and RMSE of the MLEs under different sample sizes and parameter settings. As observed, the estimates converge to the true parameter values as sample size n increases, which confirms the consistency of the proposed estimators.

In both tables, the bias of location parameter $\hat{\mu}$ is slightly negative for all cases, indicating a small systematic underestimation. The lowest bias is obtained for the location parameter, suggesting that it is estimated most accurately among all parameters. For all parameters, both bias and RMSE decrease as the sample size increases.

The estimates of $\hat{d}$ are generally more variable but exhibit the same pattern of convergence as n increases.

A comparison between Tables 3 and 4 reveals that a higher d slightly increases the bias of $\hat{d}$. Overall, the simulation confirms that the maximum likelihood estimators of the SPCN1 parameters are consistent and perform well even for small sample sizes.

6. Application

6.1. Goodness-of-fit tests

Sulewski and Stoltmann (2023) divided alternatives into nine groups according to their skewness (γ_1) and excess kurtosis ($\bar{\gamma}_2$) signs. Groups O-H are defined in Table 5. Our proposal belongs to all analysed groups except group C.

Table 6 presents parameter vectors $\theta = (0, \sigma, c, d)$ together with the corresponding values of the mean (μ_a), standard deviation (σ_a), skewness (γ_1), excess kurtosis ($\bar{\gamma}_2$) and similarity measure $M(\theta;\, \mu, \sigma)$ for selected configurations. As similarity measure M increases from 0.5 to 0.9, the parameters (σ, c, d) change smoothly, leading to different distributional shapes. The SPCN distribution is capable of producing light- and heavy-tailed, symmetric and asymmetric, as well as both unimodal and bimodal forms. Figure 9 illustrates these shapes graphically. The transition between the groups demonstrates that the SPCN family provides a coherent parametric framework for controlling skewness and kurtosis independently, while maintaining analytical tractability. These results confirm that SPCN is a highly flexible model encompassing empirical data patterns encountered in practice. We observe both unimodal and bimodal shapes.

Table 5. Groups of alternatives with signs of γ_1 and $\bar{\gamma}_2$

Group	γ_1	$\bar{\gamma}_2$
O	zero	zero
A	positive	positive
B	negative	positive

C	zero	positive
D	zero	negative
E	positive	negative
F	negative	negative
G	positive	zero
H	negative	zero

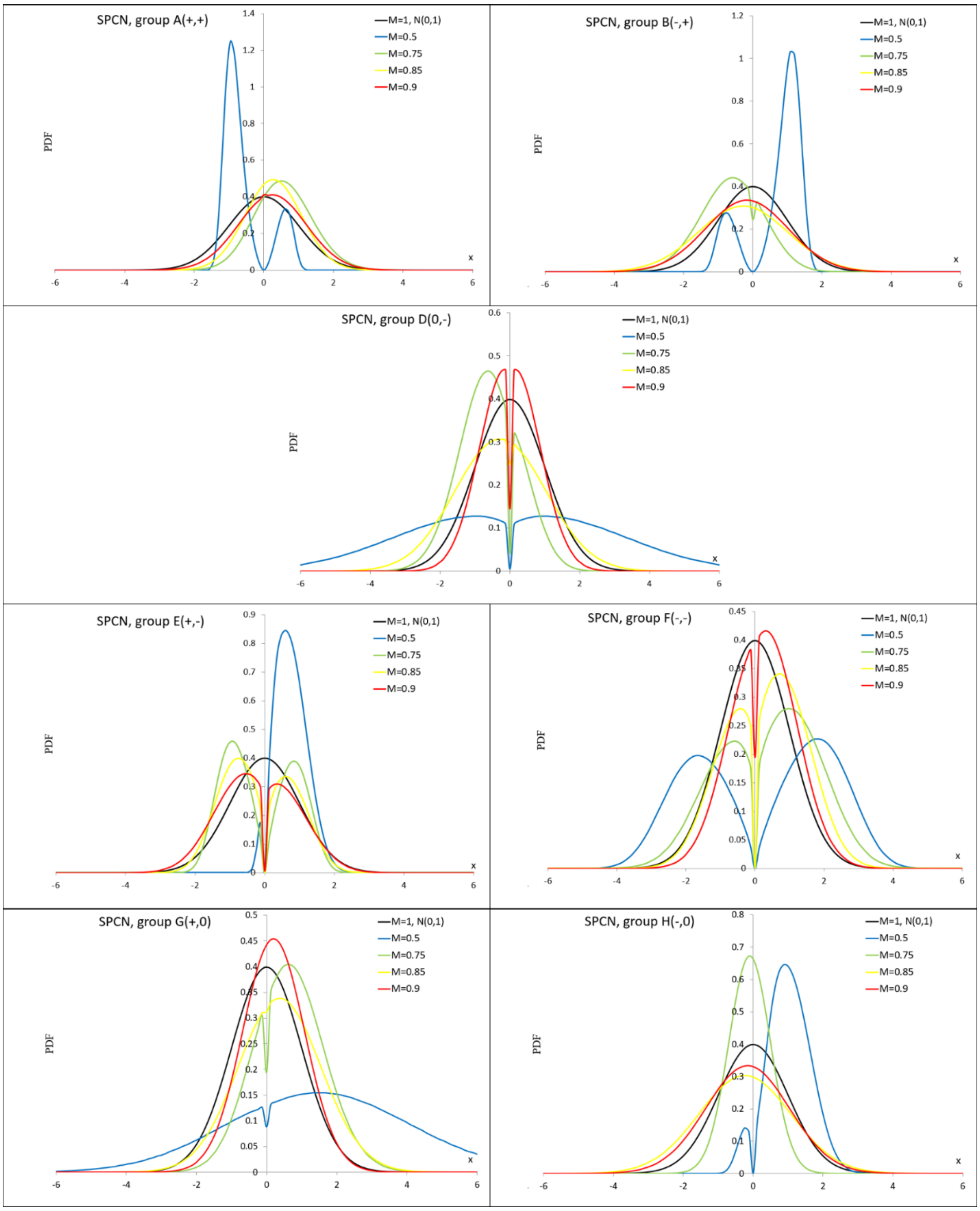
Source: authors' work.

Table 6. Vectors of SPCN parameter $\boldsymbol{\theta}$, mean μ, standard deviation σ, skewness γ_1, excess kurtosis $\bar{\gamma}_2$ and similarity measure M. Groups O-B, D-H

Group	$\boldsymbol{\theta} = (0, \sigma, c, d)$	μ	σ	$\gamma_{1,}$	$\bar{\gamma}_2$	$M(\boldsymbol{\theta}; \mu, \sigma)$
O	0,1,1,0	0	1	0	0	$M(\boldsymbol{\theta}; 0,1) = 1$
A	0,0.96,2.682, -1.141	-0.578	0.634	1.158	0.071	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,1.001,1,1	0.565	0.827	0.137	0.062	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,0.855,1,0.445	0.277	0.809	0.017	0.004	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,1.005,0.999,0.325	0.248	0.975	0.008	0.005	$M(\boldsymbol{\theta}; 0,1) = 0.9$
B	0,1.164,2.71,1.144	0.704	0.766	-1.172	0.1	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,1.102,1.012, -0.975	-0.612	0.911	-0.107	0.023	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,1.322,1, -0.277	-0.282	1.292	-0.005	0.001	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,1.202,1, -0.181	-0.171	1.189	-0.001	0.001	$M(\boldsymbol{\theta}; 0,1) = 0.9$
D	0,2.94,1.101,0	0	2.849	0	-0.343	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,1.061,1.069, -0.952	-0.573	0.865	0	-0.126	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,1.325,1.006, -0.267	-0.27	1.294	0	-0.023	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,0.817,1.038,0	0	0.806	0	-0.141	$M(\boldsymbol{\theta}; 0,1) = 0.9$
E	0,0.948,1.372,4.472	0.746	0.458	0.328	-0.1	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,1.123,1.77, -0.351	-0.122	1.001	0.13	-1.272	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,1.099,1.373, -0.32	-0.164	1.001	0.117	-0.874	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,1.156,1.126, -0.246	-0.179	1.099	0.05	-0.402	$M(\boldsymbol{\theta}; 0,1) = 0.9$
F	0,2.21,1.741,0.314	0.204	1.976	-0.108	-1.26	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,1.546,1.237,0.413	0.368	1.408	-0.13	-0.621	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,1.236,1.181,0.378	0.284	1.141	-0.1	-0.514	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,0.981,1.023,0.33	0.238	0.944	-0.01	-0.082	$M(\boldsymbol{\theta}; 0,1) = 0.9$
G	0,3.028,1.012,0.833	1.539	2.593	0.07	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,1.189,1.017,0.93	0.643	0.992	0.085	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,1.238,1.001,0.409	0.374	1.18	0.013	0	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,0.902,1,0.291	0.201	0.879	0.005	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$
H	0,1.293,1.487,2.851	0.997	0.631	-0.055	0	$M(\boldsymbol{\theta}; 0,1) = 0.5$
	0,0.602,1, -0.213	-0.1	0.594	-0.002	0	$M(\boldsymbol{\theta}; 0,1) = 0.75$
	0,1.338,1, -0.229	-0.238	1.317	-0.002	0	$M(\boldsymbol{\theta}; 0,1) = 0.85$
	0,1.206,1, -0.168	-0.159	1.196	-0.001	0	$M(\boldsymbol{\theta}; 0,1) = 0.9$

Source: authors' work.

Figure 9. PDF curves of the SPCN1 distribution for parameter values presented in Table 6



Source: authors' work.

6.2. Real data example

In this Section, we present two real data examples to demonstrate the flexibility and applicability of the SPCN1 distribution. A total of ten distributions were involved in Monte Carlo simulations.

The models selected for comparison with the SPCN1 are: ESGN3, SGN, GMNSN, FGSN3, SN1, FSCN, BABSN, FASN, SBNN, and SSGN. The PDFs of the used models are shown in Appendix 1.

The estimation of the model parameters is carried out using the maximum likelihood method. To avoid local maxima of the logarithmic likelihood function, the optimisation process is run 100 times with several different initial values widely scattered in the parameter space. AIC, BIC and HQIC were used for model comparisons. Let us recall that

$$AIC = -2l + 2p, BIC = -2l + pln(n), HQIC = -2l + 2pln(ln(n)), \tag{20}$$

where l is the log-likelihood function, n is the sample size and p is the number of model parameters. Tables 7–8 display the values of the MLEs, the information criteria (AIC, BIC and HQIC) for the analysed models. The lowest values of the information criteria are marked in bold, indicating the best-fitting model according to that criterion. It can be observed that different models perform better depending on the dataset, highlighting the flexibility and suitability of certain distributions for capturing the characteristics of the data, but the SPCN1 model achieves the best results. Plots of the estimated PDF of the analysed models are given in Figures 10 and 11. Overall, these results allow for a comprehensive assessment of model adequacy and can guide the selection of the most appropriate distribution for further statistical analysis.

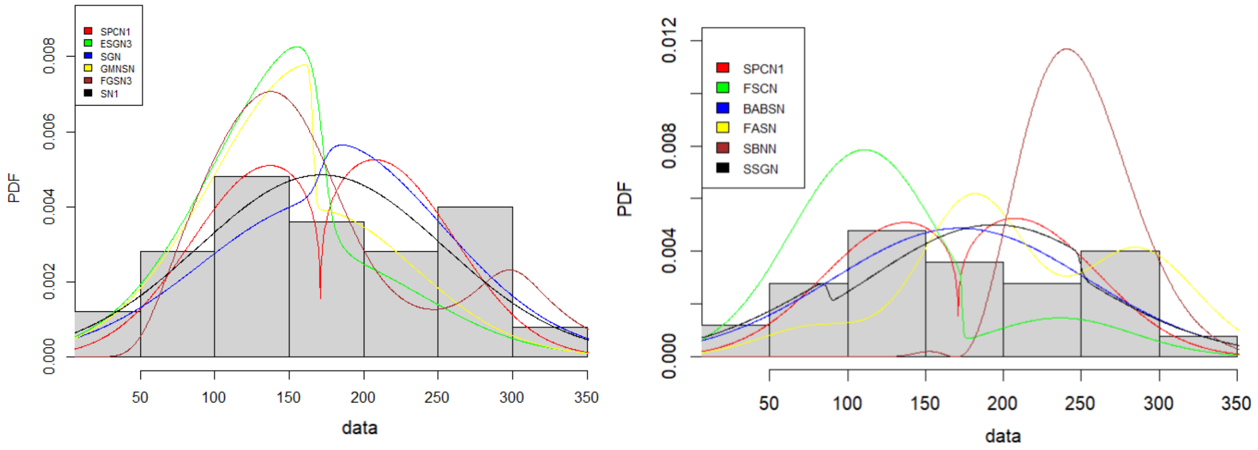
Example 1. The first dataset contains data on arrests per 100,000 residents for assault in each of the 50 US states in 1973 $(n = 50)$ (see R codes USArrests[2]). The descriptive statistics are: $\mu_a \approx 170.76$, $\sigma_a \approx 83.338$, $\gamma_1 \approx 0.227$, $\bar{\gamma}_2 \approx 1.931$.

Table 7. MLEs, AIC, BIC and HQIC (first dataset)

Model	Estimated parameters of the given model					AIC	BIC	HQIC
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	$\hat{e}$			
SPCN1	170.885	73.377	1.204	0.077		**590.689**	**598.337**	**593.601**
ESGN3	172.436	70.182	38.819	-6.098	56.707	606.892	616.452	610.532
SGN	171.026	80.465	2.235	96.356		594.334	601.982	597.246
GMNSN	165.992	67.932	-0.498	39.027		598.795	606.443	601.707
FGSN3	173.533	77.009	-1.708	0.876		603.324	610.972	606.236
SN1	171.515	82.248	41.258	-11.863		591.205	598.853	594.118
FSCN	173.546	46.995	-1.338	-47.931		608.326	615.974	611.239
BABSN	171.948	81.713	69.507	5.661		591.194	598.842	594.106
FASN	174.431	51.598	6.471	0.384		608.138	615.786	611.051
SBNN	169.436	50.301	-37.110			621.637	627.373	623.821
SSGN	168.524	83.882	46.199	0.429	62.820	596.266	605.826	599.906

Source: authors' work.

Figure 10 Estimated PDF of the analysed distributions, first dataset



Source: authors' work.

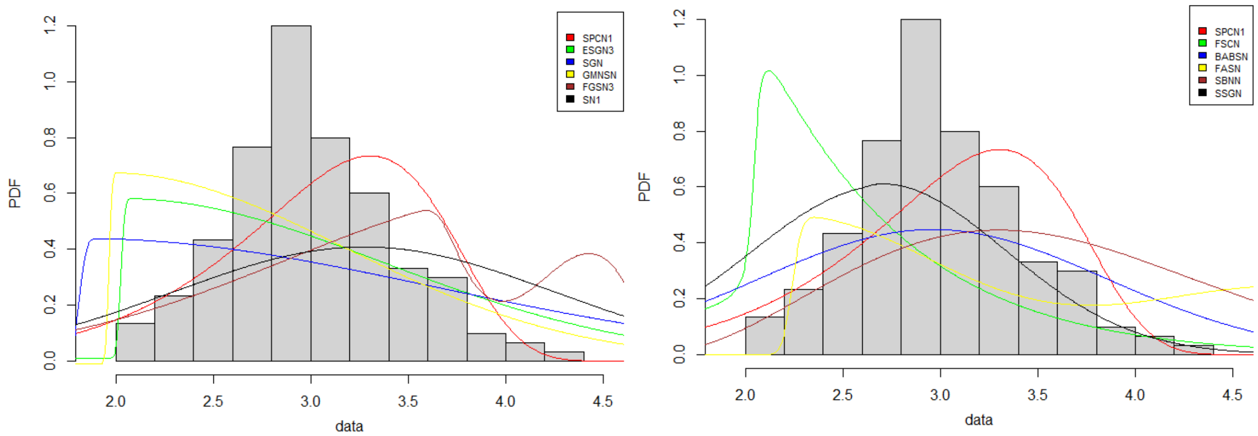
Example 2. The second set of data are measurements in centimetres of the variable sepal width for 50 flowers from each of the 3 species of iris ($n = 150$). The species are Iris setosa, versicolor, and virginica (see R codes iris[2]). The descriptive statistics are: $\mu_a \approx 3.057$, $\sigma_a \approx 0.436$, $\gamma_1 \approx 0.316$, $\bar{\gamma}_2 \approx 3.181$.

Table 8. MLEs, AIC, BIC and HQIC (second dataset)

Model	Estimated parameters					AIC	BIC	HQIC
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	$\hat{e}$			
SPCN1	-0.606	3.971	5.979	16.328		**223.498**	**235.541**	**228.391**
ESGN3	2.026	1.346	52.699	86.264	88.325	281.479	296.533	287.595
SGN	1.814	1.817	66.218	92.863		333.708	345.750	338.600
GMNSN	1.965	1.201	-71.349	98.800		269.996	282.039	274.889
FGSN3	4.212	1.314	3.213	-37.129		308.097	320.139	312.989
SN1	3.273	0.977	98.154	-26.668		313.642	325.684	318.534
FSCN	2.048	2.904	3.854	58.423		383.613	395.655	388.505
BABSN	2.962	0.891	48.565	27.686	139.243	286.486	301.540	292.602
FASN	2.241	1.207	93.347	30.114		419.210	431.253	424.103
SBNN	1.436	1.317	-23.335			298.112	307.144	301.781
SSGN	2.638	0.660	0.339	0.747	62.115	273.228	288.281	279.343

Source: authors' work.

Figure 11 Estimated PDF of the analysed distributions, second dataset



Source: authors' work.

7. Conclusions

In this paper, we propose bimodal distributions that can be used as an alternative to the other bimodal distributions in modelling bimodal-distributed data, including compound normal and Laplace distributions. The characterisation of the skew plasticising component normal I distribution is investigated. Simulation examples showed that such estimation procedures performed well. Our proposal is a very interesting alternative distribution for goodness-of-fit tests. Real data examples demonstrate that the skew plasticising component normal I distribution is a flexible, parsimonious, and competitive model that deserves to be added to the existing distributions in modelling unimodal- (see example II) and bimodal-distributed data (see example I).

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Appendix 1

Table A1. PDFs of distributions from Groups 3–7

Group no.	PDFs
3	$f_{TPSN}(x;\alpha) = \frac{2\pi}{\pi+2tan^{-1}(\alpha)}\phi(x)\Phi(\alpha\|x\|)\ (\alpha \in R).$
3	$f_{GSN1}(x;\alpha,\delta) = \begin{cases} 2\phi\left(\frac{x}{1+\delta}\right)\left[\frac{\delta}{1+\delta}+\frac{1-\delta}{1+\delta}\Phi\left(\frac{\alpha x}{1+\delta}\right)\right] & x<0 \\ 2\phi\left(\frac{x}{1-\delta}\right)\Phi\left(\frac{\alpha x}{1-\delta}\right) & x \geq 0 \end{cases}\ (\alpha \in R, \delta \in [0,1)),$
3	$f_{EESN}(x;\alpha,\delta) = \begin{cases} 2\left[\frac{\delta}{1+\delta}+\frac{1-\delta}{1+\delta}\Phi\left(\frac{\alpha x}{1+\delta}\right)\right]\phi\left(\frac{x}{1+\delta}\right), & x<0 \\ 2\Phi\left(\frac{\alpha x}{1-\delta}\right)\phi\left(\frac{x}{1-\delta}\right), & x \geq 0 \end{cases}\ (\alpha \in \mathrm{R}, \delta \in [0,1)),$
3	$f_{ESN}(x;\varepsilon) = \phi\left(\frac{x}{1+\varepsilon}\right)I(x<0)+\phi\left(\frac{x}{1-\varepsilon}\right)I(x\geq 0)\ (\|\varepsilon\|<1),$
3	$f_{FESN}(x;\alpha,\varepsilon) = \frac{1}{2-2\Phi(\delta)}\begin{cases} \phi\left(\frac{x}{1+\varepsilon}-\alpha\right) & x<0 \\ \phi\left(\frac{x}{1-\varepsilon}+\alpha\right) & x\geq 0 \end{cases}\ (\alpha \in \mathrm{R}, \|\varepsilon\|<1),$
3	$f_{STPSN}(x;\alpha,\beta) = \frac{4\pi}{\pi+2tan^{-1}(\beta)}\phi(x)\Phi(\alpha x)\Phi(\beta\|x\|)\ (\alpha,\beta \in R),$
3	$f_{GTPSN}(x;\alpha,\beta,\varepsilon) = \frac{2\pi\phi(x)\Phi_2(\alpha\|x\|,\beta\|x\|;\varepsilon)}{cos^{-1}\left(\frac{-\varepsilon-\alpha\beta}{\sqrt{1+\alpha^2}\sqrt{1+\beta^2}}\right)+tan^{-1}(\alpha)+tan^{-1}(\beta)}\ (\alpha,\beta \in R, \|\varepsilon\|<1),$ where $\Phi_2(\alpha\|x\|,\beta\|x\|;\varepsilon)$ denotes the CDF of $N_2(0{,}0{,}1{,}1,\varepsilon)$,
3	$f_{GSTPSN}(x;\alpha,\beta,\varepsilon) = c(\alpha,\beta,\rho)\phi(x)\Phi_2(\alpha x,\beta\|x\|;\varepsilon)\ (\alpha,\beta \in R, \|\varepsilon\|<1),$ where $\Phi_2(\lambda_1 x,\lambda_2\|x\|;\varepsilon)$ denotes the CDF of $N_2(0{,}0{,}1{,}1,\rho)$ and $c(\lambda_1,\lambda_2,\varepsilon) = \frac{4\pi}{cos^{-1}\left(\frac{-\varepsilon-\alpha\beta}{\sqrt{1+\alpha^2}\sqrt{1+\beta^2}}\right)+cos^{-1}\left(\frac{-\varepsilon+\alpha\beta}{\sqrt{1+\alpha^2}\sqrt{1+\beta^2}}\right)+2tan^{-1}(\beta)},$
3	$f_{GTPSN}(x;\alpha,\varepsilon) = \frac{2\pi\phi(x)}{\pi+tan^{-1}(\alpha)+tan^{-1}(\alpha\varepsilon)}\begin{cases} \Phi(\alpha x) & x<0 \\ \Phi(\alpha\varepsilon x) & x\geq 0 \end{cases}\ (\alpha \in R, \|\varepsilon\|<1),$
3	$f_{TPPN}(x,\sigma_1,\sigma_2,c) = \begin{cases} \frac{c}{\sigma_1\sqrt{2\pi}}\cdot\left(\frac{-x}{\sigma_1}\right)^{c-1} exp\left[-\frac{1}{2}\left(\frac{-x}{\sigma_1}\right)^{2c}\right] & x<0 \\ 0 & x=0 \\ \frac{c}{\sigma_2\sqrt{2\pi}}\cdot\left(\frac{x}{\sigma_2}\right)^{c-1} exp\left[-\frac{1}{2}\left(\frac{x}{\sigma_2}\right)^{2c}\right] & x>0 \end{cases}\ (\sigma_1,\sigma_2>0, c\geq 1).$
4	$f_{BN}(x;\lambda) = \left(\frac{1+\alpha\lambda}{1+\lambda}\right)\phi(x)\ (\lambda \geq 0),$
4	$f_{ASN}(x;\alpha) = \frac{(1-\alpha \mathrm{x})^2+1}{2+\alpha^2}\phi(x)\ (\alpha \in \mathrm{R}),$
4	$f_{DN}(x;\gamma) = \frac{\sqrt{\pi}\|x\|^{\gamma}}{\Gamma(\gamma+0.5)2^{\gamma}}\phi(x)\ (\gamma \geq 0),$
4	$f_{GASN}(x;\alpha,n) = \frac{(1-\alpha \mathrm{x})^{2n}+1}{2+\sum_{i=1}^{n}\binom{2n}{2i}\alpha^{2i}\prod_{j=1}^{i}(2j-1)}\phi(x)\ (\alpha \in \mathrm{R}, \mathrm{n} \in \mathrm{N}-\{0\}),$
4	$f_{BASN}(x;\alpha) = \frac{\left[(1-\alpha x)^2+1\right]^2}{3\alpha^4+8\alpha^2+4}\phi(x)\ (\alpha \in \mathrm{R}),$
4	$f_{TN}(x;\gamma) = \exp(-0.5\gamma^2)cosh(\gamma \mathrm{x})\phi(x)\ (\gamma>0),$
4	$f_{ABSN}(x;\alpha,\beta) = \frac{\left(1-\alpha x-\beta x^3\right)^2+1}{\alpha^2+15\beta^2+6\alpha\beta+2}\phi(x)\ (\alpha,\beta \in \mathrm{R}),$
4	$f_{BABSN}(x;\alpha,\beta) = \frac{\left[\left(1-\alpha x-\beta x^3\right)^2+1\right]^2}{c(\alpha,\beta)}\phi(x)\ (\alpha,\beta \in \mathrm{R}),$ where $c(\alpha,\beta) = 3\alpha^4+8\alpha^2+4+60\alpha^3\beta+12\alpha\beta(4+315\beta^2)+630\alpha^2\beta^2+15\beta^2(8+693\beta^2).$
4	$f_{FAN}(x;\gamma) = \frac{2+0.5\gamma\left[\left(x^2-1\right)^2+2\right]}{1+\gamma}\phi(x)\ (\gamma \geq 0).$

5	$f_{GN}(x;\gamma)=\frac{1}{2\gamma^{1/\gamma}\Gamma(1/\gamma)}exp\left(-\frac{\vert x\vert^{\gamma}}{\gamma}\right)(\gamma>0)$,
5	$f_{BGN}(x;\gamma)=\frac{\gamma^{(\gamma-3)/\gamma}}{2\Gamma(3/\gamma)}x^2exp\left(-\frac{\vert x\vert^{\gamma}}{\gamma}\right)(\gamma>0)$,
5	$f_{ASGN}(x;\gamma,\omega)=\frac{\gamma^{1-1/\gamma}[(1-\omega x)^2+1]}{2[\omega^2\gamma^{2/\gamma}\Gamma(3/\gamma)+2\Gamma(1/\gamma)]}exp\left(-\frac{\vert x\vert^{\gamma}}{\gamma}\right)(\gamma,\omega>0)$.
6	$f_{PN}(x;\alpha)=\alpha\phi(x)[\Phi(x)]^{\alpha-1}(\alpha>0)$,
6	$f_{GPN}(x;\alpha,\lambda)=\mathrm{k}(\alpha,\lambda)\phi(x)[\Phi(\lambda x)]^{\alpha-1}\ (\alpha>0,\lambda\in R)$,
7	$f_{PSAN}(x;\alpha,\lambda)=\alpha\phi_{\lambda}(x)[\Phi_{\lambda}(x)]^{\alpha-1}(\alpha>0,\lambda\in R)$, where $\phi_{\lambda}(x)=2\phi(x)\Phi(x)$ and $\Phi_{\lambda}(x)=\int_{-\infty}^{x}\phi_{\lambda}(t)dt$,
7	$f_{SN1}(x;\lambda_0,\lambda_1)=\Phi\left(\frac{\lambda_0}{\sqrt{1+\lambda_1^2}}\right)^{-1}\phi(x)\Phi(\lambda_0+\lambda_1x)(\lambda_0,\lambda_1\in R)$,
7	$f_{SCN}(x;\lambda)=2\phi(x)\Phi\left(\frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right)(\lambda\in R)$,
7	$f_{SGN}(x;\lambda_1,\lambda_2)=2\phi(x)\Phi\left(\frac{\lambda_1x}{\sqrt{1+\lambda_2x^2}}\right)(\lambda_1\in R,\lambda_2\geq 0)$,
7	$f_{FGSN3}(x;\lambda_1,\lambda_2)=2\phi(x)\Phi(\lambda_1x+\lambda_2x^3)\ (\lambda_1,\lambda_2\in R)$,
7	$f_{BSN}(x;\lambda,\mathrm{n})=\frac{\phi(\mathrm{x})\Phi(\lambda\mathrm{x})^n}{b_n(\lambda)}\ (\lambda\in R,n\geq 1)$, where $b_n(\lambda)=E[\Phi(\lambda\mathrm{U})^n],U\sim N(0,1)$. For $n=1,2,3$ we have closed form for $b_n(\lambda)$, i.e. $b_1(\lambda)=\frac{1}{2},b_2(\lambda)=\frac{1}{4}+\frac{1}{2\pi}sin^{-1}\left(\frac{\lambda^2}{1+\lambda^2}\right),b_3(\lambda)=\frac{1}{8}+\frac{3}{4\pi}sin^{-1}\left(\frac{\lambda^2}{1+\lambda^2}\right)$.
7	$f_{GSN2}(x;\lambda_1,\lambda_2,\rho)=\frac{2\pi\phi(x)\Phi_2(\lambda_1x,\lambda_2x;\rho)}{cos^{-1}\left(\frac{-\rho-\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)},\lambda_1,\lambda_2\in R,\vert\rho\vert<1$, where $\Phi_2(\lambda_1x,\lambda_2x;\rho)$ denotes the CDF of $N_2(0,0,1,1,\rho)$,
7	$f_{TPBSN}(x;\lambda_1,\lambda_2)=\frac{1}{c_{n,m}(\lambda_1,\lambda_2)}\phi(x)[\Phi(\lambda_1x)]^n[\Phi(\lambda_2x)]^m\ (\lambda_1,\lambda_2\in R)$, where $c_{n,m}(\lambda_1,\lambda_2)=E\{[\Phi(\lambda_1U)]^n[\Phi(\lambda_2U)]^m\},U\sim N(0,1)$,
7	$f_{GSN}(x;\lambda_1,\lambda_2,\rho)=\frac{2\pi\phi(x)\Phi_2(\lambda_1x,\lambda_2x;\rho)}{cos^{-1}\left(\frac{-\rho-\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)}\ (\lambda_1,\lambda_2\in R,\vert\rho\vert<1)$,
7	$f_{SBN}(x;\alpha,\lambda)=2\left(\frac{1+\alpha\mathrm{x}^2}{1+\alpha}\right)\phi(x)\Phi(\lambda\mathrm{x})\ (\alpha\geq 0,\lambda\in\mathrm{R})$,
7	$f_{SFN}(x;\theta,\lambda)=\frac{\phi(\vert x\vert+\theta)\Phi(\lambda x)}{1-\Phi(\theta)}\ (\theta,\lambda\in R)$,
7	$f_{ESGN1}(x;\lambda_1,\lambda_2,\lambda_3)=2\phi(x)\Phi\left(\frac{\lambda_1x}{\sqrt{\lambda_2x^2+\lambda_3x^4}}\right)(\lambda_1\in R,\lambda_2,\lambda_3>0)$,
7	$f_{ESGN2}(x;\lambda_1,\lambda_2,\lambda_3)=2\phi(x)\Phi\left(\frac{\lambda_1x}{\sqrt{1+\lambda_2x^2+\lambda_3x^4}}\right)(\lambda_1\in R,\lambda_2,\lambda_3>0)$,
7	$f_{GMNSN}(x;\alpha,\lambda)=\frac{2}{\alpha+2}\phi(x)[1+\alpha\Phi(\lambda\mathrm{x})]\ (\alpha>-2,\lambda\in R)$,
7	$f_{NSN}(x;\alpha,\beta)=\left[\frac{1}{2}-\frac{1}{\pi}tan^{-1}\left(\frac{\beta}{\sqrt{1+\alpha^2(1+\beta^2)}}\right)\right]^{-1}\phi(x)\Phi_{\beta}(\alpha x)(\alpha,\beta\in R)$, where $\Phi_{\beta}(\alpha x)=2\int_{-\infty}^{\alpha x}\phi(y)(\beta y)dy$,
7	$f_{FSGN}(x;\theta,\lambda_1,\lambda_2)=\frac{\phi(\vert x\vert+\theta)}{1-\Phi(\theta)}\Phi\left(\frac{\lambda_1x}{\sqrt{1+\lambda_2x^2}}\right)\ (\theta,\lambda_1\in R,\lambda_2>0)$,
7	$f_{FSCN}(x;\lambda,\lambda_1)=\frac{\phi(\vert x\vert+\lambda)}{1-\Phi(\lambda)}\Phi\left(\frac{\lambda_1x}{\sqrt{1+\lambda_1^2x^2}}\right)\ (\lambda,\lambda_1\in R)$,
7	$f_{ESGN3}(x;\alpha,\lambda_1,\lambda_2)=\frac{2}{\alpha+2}\phi(x)\left[1+\alpha\Phi\left(\frac{\lambda_1x}{\sqrt{1+\lambda_2x^2}}\right)\right]\ (\alpha\geq 1,\lambda_1\in R,\lambda_2\geq 0)$,
7	$f_{SSGN}(x;\alpha,\lambda_1,\lambda_2)=2\phi(x)\Phi\left(\frac{\lambda_1x}{\sqrt{1+\lambda_2\vert x\vert^{2\alpha}}}\right)\ (\alpha\neq 0,\lambda_1\in R,\lambda_2>0)$,
7	$f_{SBNN}(x;\lambda)=2x^2\phi(x)\Phi(\lambda x),\lambda\in R$,
7	$f_{BABSN}(x;\alpha,\beta)=\phi(x)\frac{\Phi\left(\beta\sqrt{\alpha^2+1}+\alpha x\right)}{\Phi(\beta)}(\alpha,\beta\in\mathrm{R})$,
7	$f_{GABSN}(x;\alpha,\beta,\lambda)=\frac{\left(1-\alpha\mathrm{x}-\beta\mathrm{x}^3\right)^2+1}{c(\alpha,\beta,\lambda)}\phi(x)\Phi(\lambda x)\ (\alpha,\beta,\lambda\in\mathrm{R})$, where $c(\alpha,\beta,\lambda)=1+3\alpha\beta-\alpha\sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}-\beta\sqrt{\frac{2}{\pi}}\frac{\lambda(3+2\lambda^2)}{(1+\lambda^2)^{1.5}}+\frac{\alpha^2}{2}+\frac{15\beta^2}{2}$,
7	$f_{FASN}(x;\alpha,\lambda)=\frac{2+0.5\alpha\left[\left(x^2-1\right)^2+2\right]}{1+\alpha}\phi(x)\Phi(\lambda x)\ (\alpha\geq 0,\lambda\in\mathrm{R})$.

Note: Functions $\phi(x)$ and $\Phi(x)$ are the PDF and CDF of the $N(0,1)$, respectively.
Source: authors' work.

Appendix 2

The following code contains R codes for the PDF, CDF, quantile, mode, k-th order moment, skewness, kurtosis, pdf of order statistics, moments of order statistics and pseudo-random number generator which is also available at github.com/PiotrSule/SPCN1.

```
library(RelDists)
library(zipfR)
library(pracma)
library(flexsurv)
library(xlsx)
library(ggamma)
library(gsl)
library(PSDistr)
library(Deriv)
library(splines)

#normalization condition
norm_cond<-function(c,d) {
 esgn1 <- function(x) dSPCN1(x,c,d)
 return (as.numeric(integrate(Vectorize(esgn1), lower = -Inf, upper = Inf)[1]))}

#CDF
library(PSDistr)
dSPCN1 <- function(x,c,d){
 return(2*dpc(x,0,1,c)*ppc(x*d,0,1,c))
}

#PDF
pSPCN1 <- function(x,c,d){
 return(integral(function(x)  dSPCN1(x,c,d),  -100,  x,  reltol  =  1e-12,  method  =
"Simpson"))
}

#quantile
qSPCN1=function(p,c,d){
 u11 = function(x,c,d) pSPCN1(x,c,d)-p
 return(uniroot(u11, c(-5,5), tol = 0.0000000001, f.lower = -5, c=c, d=d)$root)
}

#generator
rSPCN1 =function(n,c,d) {
 x=numeric(n)
 for (i in 1:n) x[i]=qSPCN1(runif(1,0,1),c,d)
 return(sort(x))
}

#ordinary moments
mSPCN1=function(k,c,d) {
 return(integral(function(x)  x^k*dSPCN1(x,c,d),  -Inf,  Inf,  reltol  =  1e-12,  method  =
"Simpson"))
}

#skewness
g1SPCN1=function(c,d){
 w1=mSPCN1(3,c,d)-3*mSPCN1(1,c,d)*mSPCN1(2,c,d)+2*mSPCN1(1,c,d)^3
```

```
 w2=mSPCN1(2,c,d)-mSPCN1(1,c,d)^2
 return(w1/w2^(1.5))
}

#kurtosis
g2SPCN1=function(c,d){
 w1=mSPCN1(4,c,d)-4*mSPCN1(1,c,d)*mSPCN1(3,c,d)+6*mSPCN1(1,c,d)^2*
 mSPCN1(2,c,d)-3*mSPCN1(1,c,d)^4
 w2=mSPCN1(2,c,d)-mSPCN1(1,c,d)^2
 return(w1/w2^2)
}

# PDF of order statistics
 dOSSPCN1=function(x,i,n,c,d) {
 return(fact(n)/fact(i-1)/fact(n-i)*dSPCN1(x,c,d)*pSPCN1(x,c,d)^(i-1)
 *(1-pSPCN1(x,c,d))^(n-i))
 }
# moments of order statistics
mOSSPCN1=function(k,i,n,c,d) {
 return(integral(function(x) x^k*dOSSPCN1(x,i,n,c,d), -Inf, Inf, reltol = 1e-12, method
= "Simpson"))
}

# Shannon entropy
sSPCN1=function(c,d){
 return(integral(function(x) -dSPCN1(x,c,d)*log(dSPCN1(x,c,d)), -Inf, Inf, reltol = 1e-
12, method = "Simpson"))
}

I11 <- function(x, c, d){
 eval(Deriv(Deriv(expression(n*log(2*c)+(c-
1)*log(abs(x))+log(dnorm(abs(x)^c,0,1))+log(porm(sign(d*x),0,1)*abs(x*d)^c)),'c'),'d')
)
}
I12 <- function(x, c, d){
 eval(Deriv(Deriv(expression(n*log(2*c)+(c-
1)*log(abs(x))+log(dnorm(abs(x)^c,0,1))+log(porm(sign(d*x),0,1)*abs(x*d)^c)),'c'),'d')
)
}
I21 <- function(x, c, d) return(I12(x,c,d))
I22 <- function(x, c, d){
 eval(Deriv(Deriv(expression(n*log(2*c)+(c-
1)*log(abs(x))+log(dnorm(abs(x)^c,0,1))+log(porm(sign(d*x),0,1)*abs(x*d)^c)),'d'),'d')
)
}

# Fisher Information Matrix
fimSPCN1=function(c,d,xg){
 FIM=numeric(4)
 FIM[1]=-integral(function(x) I11(x,c,d)*dSPCN1(x,c,d), -xg, xg, reltol = 1e-9, method
= "Kronrod")
 FIM[2]=-integral(function(x) I12(x,c,d)*dSPCN1(x,c,d), -xg, xg, reltol = 1e-9, method
= "Kronrod")
 FIM[3]=FIM[2]
 FIM[4]=-integral(function(x) I22(x,c,d)*dSPCN1(x,c,d), -xg, xg, reltol = 1e-9, method
= "Kronrod")
 return(FIM)
}

#Hessian Matrix
hmSPCN1=function(c,d){
 HM=numeric(4)
 HM[1]=eval(Deriv(Deriv(expression(2*c*abs(x)^(c-1)*
```

```
 dnorm(abs(x)^c,0,1)*pnorm(sign(d*x)*abs(x*d)^c,0,1)),'c'),'c'))
 HM[2]=eval(Deriv(Deriv(expression(2*c*abs(x)^(c-1)*
 dnorm(abs(x)^c,0,1)*pnorm(sign(d*x)*abs(x*d)^c,0,1)),'c'),'d'))
 HM[3]=HM[2]
 HM[4]=eval(Deriv(Deriv(expression(2*c*abs(x)^(c-1)*
 dnorm(abs(x)^c,0,1)*pnorm(sign(d*x)*abs(x*d)^c,0,1)),'d'),'d'))
 return(HM)
}
```